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Convergence rates to traveling waves for viscous conservation laws with dispersion

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Abstract

In this paper we investigate the time decay rates of perturbations of the traveling waves for viscous conservation laws with dispersion. The convergence rates in time to traveling waves are obtained when the initial data have different asymptotic limits at the far fields $\pm\infty$. This improves previous results on decay rates.

1. Introduction

Consider viscous conservation laws with dispersion

$$u_t + f(u)_x + u_{xxx} - \mu u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

where the flux function $f : \mathbb{R} \to \mathbb{R}$ is smooth, and the constant $\mu > 0$ is the viscosity parameter. The initial data

$$u(x,0) = u_0(x) \tag{1.2}$$

are assumed to satisfy $u_0(x) \to u_\pm$ as $x \to \pm\infty$.

We consider monotone traveling wave solutions $\phi(x - st)$ to Eq. (1.1). Such a profile ϕ satisfies the following O.D.E. [11, Chapter 2]

$$\phi_{zzz} - \mu\phi_{zz} + (f(\phi) - s\phi)_z = 0, \quad z = x - st. \tag{1.3}$$

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The far-field states $u_\pm \in \mathbb{R}$ and the wave speed $s \in \mathbb{R}$ satisfy the Rankine–Hugoniot condition

$$f(u_+) - f(u_-) - s(u_+ - u_-) = 0 \tag{1.4}$$

and the Oleinik shock condition [10]

$$f(u) - f(u_-) - s(u - u_-) \begin{cases} < 0 & \text{if } u_+ < u < u_-, \\ > 0 & \text{if } u_- < u < u_+. \end{cases} \tag{1.5}$$

If $f'' > 0$ then (1.5) implies the Lax shock condition

$$f'(u_+) < s < f'(u_-). \tag{1.6}$$

For the case $f(u) = \frac{u^2}{2}$, Eq. (1.1) reduces to the Korteweg-de Vries–Burgers equation

$$u_t + uu_x + u_{xxx} - \mu u_{xx} = 0. \tag{1.7}$$

Grad and Hu [5], as well as Bona and Schonbek [1] showed that (1.7) admits traveling wave solutions $\phi(x - st)$ connecting u_- to u_+, which are monotone if $\mu \gg 1$. Moreover, in obtaining solutions to this problem one can look for solutions which are the sum of a traveling wave and a perturbation. It has been shown that this traveling wave is asymptotically stable for small perturbations by Bona et al. [2]. Another proof of the stability was given by Pego [12] based on the energy method due to Goodman [4]. For the case $f(u) = u^3$, Dodd [3] showed that certain traveling wave solutions are asymptotically stable with respect to a weighted norm as perturbations convect away from the wave profile in the direction indicated by the group velocity associated with the linearized perturbation equation. Warnecke and Pan [14] consider a more general equation of form (1.1) and obtained corresponding stability results. Recently, pointwise methods are also developed to investigate the stability of the traveling waves for equations of type (1.1) by Howard and Zumbrun [6].

In further work on the Korteweg-de Vries–Burgers equation (1.7), Rajopadhye [13] showed that if the perturbation lies in a suitable weighted class, then it decays: the L^2-norm of the perturbation decays at the rate of $(1+t)^{-\frac{1}{4}}$ and the L^2-norm of the first-order derivative decays at the rate of $(1+t)^{-\frac{3}{4}}$ as $t \to \infty$. Recently, Nishihara and Rajopadhye [9] generalized the above result and obtained that the weighted L^2-norm of the perturbation decays at the rate of $(1+t)^{-\frac{k-\varepsilon}{2}}$, provided the initial perturbation decays at the rate $|x|^{-k}$ in space, where $\varepsilon = 0$ if k is an integer and $\varepsilon > 0$ otherwise.

In this paper we are concerned about the more general equation (1.1) and get the optimal decay rate $(1+t)^{-\frac{k}{2}}$ for any $k \geqslant 0$, which improves the previous result for

KdVB equation obtained in [9]. Moreover, due to the presence of dispersion, the arguments are much more involved when we replace $\frac{u^2}{2}$ by general convex flux function $f(u)$.

This paper is organized as follows: in Section 2 we state some results obtained in [14], which will be used in this work. In Section 3, we establish the L^2 decay rates of the perturbations of traveling wave solutions. In Section 4 we study the decay rates for higher order derivatives.

Notation. We denote by C a generic constant. Moreover, in the inequalities that follow, the constant C can change from one line to the next. For function spaces, L^2 denotes the space of square integrable functions on $\mathbb{R}$ with the norm

$$||f|| = \left(\int_{\mathbb{R}} |f(x)|^2\, dx\right)^{\frac{1}{2}}.$$

The usual lth order norm of the Sobolev space H^l, $l \geqslant 0$ is denoted by

$$||f||_l = \left(\sum_{j=0}^{l} ||\partial_x^j f||^2\right)^{\frac{1}{2}}.$$

For a weight function w, L^2_w denotes the space of measurable functions f satisfying $\sqrt{w} f \in L^2$ with the norm

$$|f|_w = \left(\int_{\mathbb{R}} w(x)|f(x)|^2\, dx\right)^{\frac{1}{2}}.$$

When $w(x) = \langle x \rangle^{\alpha} = (1+x^2)^{\frac{\alpha}{2}}$, we write $L^2_w = L^2_\alpha$ and $|\cdot|_w = |\cdot|_\alpha$.

Let T be a positive constant and B a Banach space. Here, for any natural number $k \geqslant 0$, $C^k([0,T];B)$ denotes the space of B-valued k-times continuously differentiable functions on $[0,T]$. The space of B-valued L^2 functions on $[0,T]$ is denoted by $L^2([0,T];B)$. The corresponding spaces of B-valued functions on $[0,\infty)$ are denoted by $C^k([0,\infty);B)$ and $L^2([0,\infty);B)$, respectively.

2. Asymptotic stability of traveling waves

For completeness, in this section we state the stability results of traveling wave solutions, which are the basis of the study for the decay rates of perturbations.

Assume that ϕ is a monotone traveling wave solution satisfying the O.D.E. (1.3) and that the initial perturbation $u_0 - \phi$ is integrable over $\mathbb{R}$. It is known that solutions with the initial data near ϕ will typically approach a translate of ϕ rather than ϕ itself, but the shift can usually be determined by the mass carried by the initial perturbation. We define x_0 by

$$\int_{\mathbb{R}} [u_0(x) - \phi(x)]\, dx = x_0(u_+ - u_-), \tag{2.1}$$

where u_0 are the initial data given in (1.2). Observe that

$$\int_{\mathbb{R}} [\phi(x + x_0) - \phi(x)]\, dx = x_0(u_+ - u_-).$$

This follows from the fact that the left-hand side regarded as the function of x_0 is differentiable, its differention is $u_+ - u_-$ and it vanishes at $x_0 = 0$. Therefore, relation (2.1) is equivalent to

$$\int_{\mathbb{R}} [u_0(x) - \phi(x + x_0)]\, dx = 0.$$

Without loss of generality let us assume that $x_0 = 0$, i.e.,

$$\int_{\mathbb{R}} [u_0(x) - \phi(x)]\, dx = 0. \tag{2.2}$$

Note that for given initial data u_0 condition (2.2) selects a unique traveling wave ϕ from the whole family of shifted traveling waves satisfying (1.3).

We define

$$w_0(x) \equiv \int_{-\infty}^{x} [u_0(y) - \phi(y)]\, dy. \tag{2.3}$$

Then stability results obtained in [14] are given as follows.

Theorem 2.1. *Assume that the Rankine–Hugoniot condition* (1.4) *and the Lax shock condition* (1.6) *hold. Let* ϕ *be a monotonically decreasing traveling wave solution of* (1.1) *uniquely determined by* (2.2). *Suppose* $u_0 - \phi$ *is integrable over* $\mathbb{R}$ *and the primitive of this difference satisfies* $w_0 \in H^3(\mathbb{R})$. *Then there exists a constant* $\varepsilon > 0$ *such that if* $||w_0||_3 < \varepsilon$ *the Cauchy problem* (1.1), (1.2) *has a unique global solution* u *satisfying*

$$u - \phi \in C([0, \infty); H^2(\mathbb{R})) \cap L^2([0, \infty); H^3(\mathbb{R}))$$

and

$$\sup_{x \in \mathbb{R}} |u(x,t) - \phi(x - st)| \to 0 \quad \text{as } t \to \infty.$$

Proof. To prove this theorem, we set

$$w(x,t) = \int_{-\infty}^{x} [u(y+st,t) - \phi(y)]\,dy,$$

which satisfies $\lim_{x\to\pm\infty} w(x,t) = 0$ by (2.2) and the conservation form of the equation (1.1). Now we decompose the solution to equation (1.1) into the form

$$u(x,t) = \phi(z) + w_z(z,t), \quad z = x - st$$

where ϕ is a monotone traveling wave solution to (1.1) and w_z denotes the perturbation. Upon substitution into (1.1) problem (1.1), (1.2) is reduced to

$$w_t - sw_z + f(\phi + w_z) - f(\phi) + w_{zzz} - \mu w_{zz} = 0 \tag{2.4}$$

with

$$w(z,0) = w_0(z) \equiv \int_{-\infty}^{z} [u_0(y) - \phi(y)]\,dy. \tag{2.5}$$

Using

$$Q(u) \equiv f(u) - f(u_-) - s(u - u_-) \tag{2.6}$$

gives $Q'(\phi) = f'(\phi) - s$, where s is the shock speed depending on u_+ and u_-, see (1.4). Eq. (2.4) may be rewritten as

$$w_t + Q'(\phi)w_z + w_{zzz} - \mu w_{zz} = F(\phi, w_z) \tag{2.7}$$

with

$$F(\phi, w_z) = -[f(\phi + w_z) - f(\phi) - f'(\phi)w_z].$$

We note that

$$F(\phi, w_z) = -f''(\phi + \theta w_z)w_z^2 \tag{2.8}$$

for suitable $\theta(z,t) \in]0,1[$. This will be used later.

Theorem 2.1 is the consequence of the following result.

Theorem 2.2. *Suppose $w_0 \in H^3(\mathbb{R})$. Then there exists a positive constant ε such that if $||w_0||_3 < \varepsilon$, we obtain a solution w to* (2.7), (2.5) *in $C([0,\infty), H^3(\mathbb{R}))$ satisfying*

$$||w(t)||_3^2 + \int_0^t ||w_z(\tau)||_3^2\,d\tau \leqslant C||w_0||_3^2, \tag{2.9}$$

and moreover

$$||w_z(,t)||_2 \to 0 \quad as\ t \to \infty.$$

This global existence and decay for w was derived from the Local Existence Theorem for w combined with the a priori estimate obtained in [14]. $\square$

3. L^2 decay rate

In this section we establish the L^2 decay rates. We have the theorem below.

Theorem 3.1. *Let w be the solution to* (2.7) *obtained in Theorem* 2.2. *If $w_0 \in L^2_\alpha(\mathbb{R})$ for some $\alpha \geqslant 0$, then for $0 \leqslant k \leqslant \alpha$ we have*

$$|w(t)|_{\alpha-k} \leqslant C|w_0|_\alpha (1+t)^{-\frac{k}{2}}. \tag{3.1}$$

Note that $|w(\cdot)|_{\alpha-k} = ||\langle z-z^*\rangle^{\frac{\alpha-k}{2}} w(t)||$, where z^* is a fixed constant satisfying $f'(\phi(z^*)) = s$ and $\langle z-z^*\rangle = \sqrt{1+(z-z^*)^2}$.

Proof. Let $K = (1+t)^\gamma \langle z-z^*\rangle^\beta$. Multiplying (2.7) by $2Kw$, we have

$$2Kww_t + 2Q'Kww_z + 2Kww_{zzz} - 2\mu Kww_{zz} = 2KwF.$$

Using

$$H \coloneqq 2Kww_{zz} - Kw_z^2 - 2K_zww_z + K_{zz}w^2 + \mu(K_zw^2 - 2Kww_z),$$

we obtain

$$\begin{aligned}
&[Kw^2]_t - K_tw^2 + 2\mu Kw_z^2 + 3K_zw_z^2 + [KQ'](w^2)_z \\
&\quad - \{K_{zzz} + \mu K_{zz}\}w^2 + H_z = -2KwF.
\end{aligned} \tag{3.2}$$

Integrating (3.2) over $\mathbb{R}$ and using $K = (1+t)^\gamma \langle z-z^*\rangle^\beta$ we obtain

$$\begin{aligned}
&[(1+t)^\gamma |w(t)|_\beta^2]_t - \int_{\mathbb{R}} [KQ']_z w^2\, dz + 2\mu \int_{\mathbb{R}} Kw_z^2\, dz \\
&\quad \leqslant \gamma(1+t)^{\gamma-1}|w(t)|_\beta^2 + \left| \int_{\mathbb{R}} (K_{zzz} + \mu K_{zz})w^2\, dz \right|
\end{aligned}$$

$$+3\int_{\mathbb{R}}|K_z|w_z^2\,dz+2\int_{\mathbb{R}}|KwF|\,dz. \tag{3.3}$$

In order to finish the proof of Theorem 3.1, first let us prove the following lemmas.

Lemma 3.2. *For $z^*\in\mathbb{R}$ satisfying $f'(\phi(z^*))=s$, there exists a positive constant C_0 such that*

$$-[KQ']_z \geqslant C_0\beta(1+t)^\gamma\langle z-z^*\rangle^{\beta-1},\quad \beta\in[0,\alpha]\ \textit{with fixed}\ \alpha\geqslant 0. \tag{3.4}$$

Proof. Noting the definition of K and Q, one gets

$$-[KQ']_z=(1+t)^\gamma\langle z-z^*\rangle^{\beta-1}A_\beta(z)$$

with

$$A_\beta(z)=I_1(z)+I_2(z),$$

where

$$I_1(z)=-\beta(z-z^*)\langle z-z^*\rangle^{-1}[f'(\phi)-s],$$
$$I_2(z)=-\langle z-z^*\rangle f''(\phi)\phi_z.$$

Obviously $I_2(z)\geqslant 0$ for all $z\in\mathbb{R}$. It will be proved that $I_1(z)\geqslant 0$ for all $z\in\mathbb{R}$.

First, one can choose $z^*\in\mathbb{R}$ such that $f'(\phi(z^*))=s$. In fact, from the inequality $f''(u)>0$, i.e. f is convex, one can deduce that $f'(u)$ increases strictly in $u\in[u_+,u_-]$. Therefore, by the Lax shock condition we can choose a unique state $u^*\in(u_+,u_-)$ such that $f'(u^*)=s$ since $f'(u_+)<s<f'(u_-)$. Moreover, the traveling wave solution ϕ is strictly decreasing in $z\in\mathbb{R}$ by Theorem 2.3.1 [11]. Therefore, there exists uniquely a number $z^*\in\mathbb{R}$ such that $\phi(z^*)=u^*$, i.e. $f'(\phi(z^*))=s$.

Next, we can prove that $I_1(z)\geqslant 0$ for all $z\in R$. In fact, by the mean value theorem one obtains a suitable $\bar{z}$ between z and z^* such that

$$\begin{aligned}I_1(z)&=-\beta(z-z^*)\langle z-z^*\rangle^{-1}[f'(\phi)-f'(\phi(z^*))]\\&=-\beta\frac{(z-z^*)^2}{\langle z-z^*\rangle}f''(\phi(\bar{z}))\phi_z(\bar{z})\geqslant 0.\end{aligned}$$

Function $I_1(z)=0$ if and only if $z=z^*$.

Because $I_i(z)\geqslant 0$, $i=1,2$,

$$A_\beta(z)\geqslant I_1(z)\quad\text{and}\quad A_\beta(z)\geqslant I_2(z). \tag{3.5}$$

Finally, we want to prove that there exists a positive constant C_0 such that

$$A_\beta(z) \geqslant C_0\beta, \quad \beta \in [0, \alpha] \quad \text{for any given } \alpha \geqslant 0 \tag{3.6}$$

for any $z \in \mathbb{R}$.

In order to do this, we set $\mathbb{R} = \{z : |z - z^*| < \delta\} \cup \{z : |z - z^*| \geqslant \delta\}$, here $\delta > 0$ being suitably small that if $z \in \{z : |z - z^*| < \delta\}$, then

$$|\phi_z(z)| \geqslant \frac{|\phi_z(z^*)|}{2}.$$

This gives

$$\begin{aligned} -\langle z - z^* \rangle f''(\phi)\phi_z &\geqslant \min_{\phi \in [u_+, u_-]} f''(\phi)\frac{|\phi_z(z^*)|}{2} \\ &= \min_{\phi \in [u_+, u_-]} f''(\phi)\frac{|\phi_z(z^*)|}{2}\frac{1}{\alpha}\alpha \\ &\geqslant C_2\beta, \end{aligned} \tag{3.7}$$

where $C_2 = \min_{\phi \in [u_+, u_-]} f''(\phi)\frac{|\phi_z(z^*)|}{2}/\alpha$.

In the set $\{z : |z - z^*| \geqslant \delta\}$ we consider I_1. By the Lax shock condition, we have

$$\lim_{z \to \pm\infty} I_1(z) = \beta|f'(u_\pm) - s| > 0.$$

This implies that there exists a constant $C_1 > 0$ such that

$$I_1(z) \geqslant C_1\beta \quad \text{for } |z - z^*| \geqslant \delta. \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we have proved (3.6) and thereby (3.4). $\square$

Lemma 3.3. *For* $0 < t \leqslant T$, *we have the following estimate*:

$$\begin{aligned} &\left| \int_{\mathbb{R}} (K_{zzz} + \mu K_{zz}) w^2 \, dz \right| + 3 \int_{\mathbb{R}} |K_z| w_z^2 \, dz \\ &\quad \leqslant \frac{C_0\beta}{2}(1 + t)^\gamma |w(t)|_{\beta-1}^2 \\ &\qquad + 2C\beta(1 + t)^\gamma \int_{\mathbb{R}} (\langle z - z^* \rangle^{\beta-3} + \langle z - z^* \rangle^{\beta-1}) w_z^2 \, dz. \end{aligned} \tag{3.9}$$

Proof. By integration by parts one gets

$$
\begin{aligned}
&\int_{\mathbb{R}} (K_{zzz} + \mu K_{zz}) w^2 \, dz \\
&\quad = -2 \int_{\mathbb{R}} (K_{zz} + \mu K_z) w w_z \, dz \\
&\quad = 2\beta(1+t)^\gamma \int_{\mathbb{R}} \{ \langle z - z^* \rangle^{\beta-2} + (\beta-2)(z-z^*)^2 \langle z - z^* \rangle^{\beta-4} \\
&\qquad + \mu (z - z^*) \langle z - z^* \rangle^{\beta-2} \} w w_z \, dz.
\end{aligned}
$$

We set $a = w$ and $b = v w_z$ with

$$
v \coloneqq \langle z - z^* \rangle^{\beta-2} + (\beta-2)(z-z^*)^2 \langle z - z^* \rangle^{\beta-4} + \mu (z-z^*) \langle z - z^* \rangle^{\beta-2}.
$$

Choosing $\varepsilon = \frac{C_0}{4} \langle z - z^* \rangle^{\beta-1}$ and using the inequality $ab \leqslant \varepsilon a^2 + \frac{b^2}{4\varepsilon}$ we get

$$
\begin{aligned}
\left| \int_{\mathbb{R}} (K_{zzz} + \mu K_{zz}) w^2 \, dz \right| &\leqslant 2\beta(1+t)^\gamma \int_{\mathbb{R}} \varepsilon w^2 \, dz + 2\beta(1+t)^\gamma \int_{\mathbb{R}} \frac{v^2}{4\varepsilon} w_z^2 \, dz \\
&\leqslant \frac{C_0 \beta}{2} (1+t)^\gamma \int_{\mathbb{R}} \langle z - z^* \rangle^{\beta-1} w^2 \, dz \\
&\quad + \frac{2\beta}{C_0} (1+t)^\gamma \int_{\mathbb{R}} \frac{v^2}{\langle z - z^* \rangle^{\beta-1}} w_z^2 \, dz.
\end{aligned}
$$

Now using the inequality $(a+b+c)^2 \leqslant 3a^2 + 3b^2 + 3c^2$ and $(z-z^*)^2 \leqslant \langle z - z^* \rangle^2$, we can estimate

$$
v^2 \leqslant C(\langle z - z^* \rangle^{2\beta-4} + \langle z - z^* \rangle^{2\beta-2}).
$$

Thus

$$
\begin{aligned}
\left| \int_{\mathbb{R}} (K_{zzz} + \mu K_{zz}) w^2 \, dz \right| &\leqslant \frac{C_0 \beta}{2} (1+t)^\gamma |w(t)|_{\beta-1}^2 \, dz + C\beta(1+t)^\gamma \\
&\quad \times \int_{\mathbb{R}} (\langle z - z^* \rangle^{\beta-3} + \langle z - z^* \rangle^{\beta-1}) w_z^2 \, dz. \qquad (3.10)
\end{aligned}
$$

In addition,

$$
\begin{aligned}
\int_{\mathbb{R}} |K_z| w_z^2 \, dz &= \beta(1+t)^\gamma \int_{\mathbb{R}} |z - z^*| \langle z - z^* \rangle^{\beta-2} w_z^2 \, dz \\
&\leqslant \beta(1+t)^\gamma \int_{\mathbb{R}} \langle z - z^* \rangle^{\beta-1} w_z^2 \, dz. \qquad (3.11)
\end{aligned}
$$

Combining (3.10) with (3.11) we have finished the proof of the lemma. $\square$

Lemma 3.4. *Let $\alpha \geqslant 0$ be fixed and $\beta \leqslant \alpha$, then we have the estimate*

$$C\beta \int_{\mathbb{R}} (\langle z - z^* \rangle^{\beta-3} + \langle z - z^* \rangle^{\beta-1}) w_z^2 \, dz \leqslant \frac{\mu}{2} |w_z|_\beta^2 + C_r \beta ||w_z||^2, \tag{3.12}$$

where $C_r = C(r^{|\alpha-1|} + r^{|\alpha-3|})$ for some positive number r which satisfies $\frac{2C\alpha}{r} \leqslant \frac{\mu}{2}$.

Proof. Let $\mathbb{R} = B \cup B^c$, where $B = \{z : |z - z^*| \leqslant r\}$ and B^c is the usual complement of B. One then has

$$\begin{aligned} C\beta \int_{\mathbb{R}} &(\langle z - z^* \rangle^{\beta-1} + \langle z - z^* \rangle^{\beta-3}) w_z^2 \, dz \\ &= C\beta \int_{B^c} (\langle z - z^* \rangle^{\beta-1} + \langle z - z^* \rangle^{\beta-3}) w_z^2 \, dz \\ &\quad + \int_{B} (\langle z - z^* \rangle^{\beta-1} + \langle z - z^* \rangle^{\beta-3}) w_z^2 \, dz \\ &\leqslant \frac{2C\beta}{r} \int_{B^c} \langle z - z^* \rangle^{\beta} w_z^2 \, dz + C_r \beta \int_B w_z^2 \, dz \\ &\leqslant \frac{\mu}{2} \int_{\mathbb{R}} \langle z - z^* \rangle^{\beta} w_z^2 \, dz + C_r \beta \int_{\mathbb{R}} w_z^2 \, dz. \end{aligned}$$

This completes the proof of the lemma. □

Combining Lemma 3.3 with Lemma 3.4 gives the following corollary:

Corollary 3.5. *Let $\alpha \geqslant 0$ be fixed and $\beta \leqslant \alpha$. For $0 < t \leqslant T$, we have the following estimate*:

$$\begin{aligned} &\left| \int_{\mathbb{R}} (K_{zzz} + \mu K_{zz}) w^2 \, dz \right| + 3 \int_{\mathbb{R}} |K_z| w_z^2 \, dz \\ &\quad \leqslant \frac{C_0 \beta}{2} (1+t)^\gamma |w|_{\beta-1}^2 + \mu (1+t)^\gamma |w_z|_\beta^2 + 2 C_r \beta ||w_z||^2. \end{aligned} \tag{3.13}$$

Finally, the nonlinear term is bounded using (2.8) by

$$2 \int_{\mathbb{R}} |KwF| \, dz \leqslant C ||w||_\infty (1+t)^\gamma |w_z|_\beta^2. \tag{3.14}$$

Due to (2.9) in combination with the Sobolev lemma, the norm $||w||_\infty$ is small if $||w_0||_3$ is sufficiently small. Therefore, the left-hand side of (3.14) can be absorbed by the third term on the left-hand side of (3.3). By (3.3), (3.4), (3.13) and (3.14) we obtain the inequality

$$[(1+t)^\gamma |w(t)|_\beta^2]_t + \frac{C_0\beta}{2}(1+t)^\gamma |w(t)|_{\beta-1}^2 + \frac{\mu}{2}(1+t)^\gamma |w_z(t)|_\beta^2 \leqslant \gamma(1+t)^{\gamma-1}|w(t)|_\beta^2 + 2C_r\beta(1+t)^\gamma ||w_z(t)||^2 \tag{3.15}$$

for any $\beta \leqslant \alpha$.

We set

$$C = \frac{\max\{1, 2C_r\}}{\min\{1, \frac{C_0}{2}, \frac{\mu}{2}\}}.$$

Integrating the inequality (3.15) over $\tau \in [0, t]$ one gets

$$(1+t)^\gamma |w(t)|_\beta^2 + \int_0^t [\beta(1+\tau)^\gamma |w(\tau)|_{\beta-1}^2 + (1+\tau)^\gamma |w_z(\tau)|_\beta^2]\, d\tau \leqslant C\left[|w_0|_\beta^2 + \int_0^t [\gamma(1+\tau)^{\gamma-1}|w(\tau)|_\beta^2 + \beta(1+\tau)^\gamma ||w_z(\tau)||^2]\, d\tau\right]. \tag{3.16}$$

As in Kawashima and Matsumura [7] or in Matsumura and Nishihara [8], by induction we have

Lemma 3.6. *For $k = 0, 1, \ldots, [\alpha]$, the inequality*

$$(H_k) \quad (1+t)^k |w(t)|_{\alpha-k}^2 + \int_0^t [(\alpha-k)(1+\tau)^k |w(\tau)|_{\alpha-k-1}^2 + (1+\tau)^k |w_z|_{\alpha-k}^2]\, d\tau \leqslant C|w_0|_\alpha^2 \tag{3.17}$$

holds.

For convenience of the readers, we give an improved argument as following.

Proof. *Step* 1. We prove (H_0).

Taking $\beta = 0, \gamma = 0$ in (3.16), one gets

$$||w(t)||^2 + \int_0^t ||w_z(\tau)||^2\, d\tau \leqslant C||w_0||^2. \tag{3.18}$$

Taking $\beta = \alpha, \gamma = 0$ in (3.16) and using (3.18), one gets

$$|w(t)|_\alpha^2 + \int_0^t [\alpha|w(\tau)|_{\alpha-1}^2 + |w_z(\tau)|_\alpha^2]\, d\tau \leqslant C\left[|w_0|_\alpha^2 + \alpha\int_0^t ||w_z(\tau)||^2\, d\tau\right] \leqslant C|w_0|_\alpha^2. \tag{3.19}$$

This is (H_0).

Step 2. Assume (H_{k-1}) holds, i.e.

$$(1+t)^{k-1}|w(t)|^2_{\alpha-(k-1)} + \int_0^t [(\alpha-k+1)(1+\tau)^{k-1}|w(\tau)|^2_{\alpha-k} + (1+\tau)^{k-1}|w_z|^2_{\alpha-(k-1)}]\, d\tau \leqslant C|w_0|^2_\alpha, \tag{3.20}$$

we prove (H_k).

Taking $k \leqslant \alpha$. From (3.20) we have

$$\int_0^t (1+\tau)^{k-1}||w(\tau)||^2\, d\tau \leqslant \int_0^t (1+\tau)^{k-1}|w(\tau)|^2_{\alpha-k}\, d\tau \leqslant C|w_0|^2_\alpha. \tag{3.21}$$

Let $\beta = 0, \gamma = k$ in (3.16) and using (3.21) we have

$$\begin{aligned}(1+t)^k||w(t)||^2 &+ \int_0^t (1+\tau)^k||w_z(\tau)||^2\, d\tau \\ &\leqslant C\left[||w_0||^2 + \int_0^t k(1+\tau)^{k-1}||w(\tau)||^2\, d\tau\right] \\ &\leqslant C|w_0|^2_\alpha,\end{aligned} \tag{3.22}$$

Let $\beta = \alpha - k, \gamma = k$ in (3.16) and using (3.20), (3.22), we have

$$\begin{aligned}&(1+t)^k|w(t)|^2_{\alpha-k} + \int_0^t [(\alpha-k)(1+\tau)^k|w(\tau)|^2_{\alpha-k-1} + (1+\tau)^k|w_z(\tau)|^2_{\alpha-k}]\, d\tau \\ &\quad\leqslant C\left[|w_0|^2_{\alpha-k} + \int_0^t [k(1+\tau)^{k-1}|w(\tau)|^2_{\alpha-k} + (\alpha-k)(1+\tau)^k||w_z(\tau)||^2]\, d\tau\right] \\ &\quad\leqslant C|w_0|^2_\alpha,\end{aligned} \tag{3.23}$$

which is (H_k). By induction argument, (H_k) holds for any k range from 0 to $[\alpha]$. $\square$

Inequality (3.17) proves Theorem 3.1 when k is an integer. $\square$

Next we consider the case when both α and k are not integers.

Lemma 3.7. *For any $\varepsilon > 0$, we have*

$$(1+t)^\alpha||w(t)||^2 + (1+t)^{-\varepsilon}\int_0^t (1+\tau)^{\alpha+\varepsilon}||w_z(\tau)||^2\, d\tau \leqslant C_\varepsilon|w_0|^2_\alpha$$

with $C_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Proof. Taking $k = [\alpha]$ in (3.17) yields

$$(1+t)^{[\alpha]}|w(t)|^2_{\alpha-[\alpha]} + \int_0^t (1+\tau)^{[\alpha]}|w(\tau)|^2_{\alpha-[\alpha]-1}\, d\tau \leqslant C|w_0|^2_\alpha. \tag{3.24}$$

In (3.16) we take $\beta = 0$, $\gamma = \alpha + \varepsilon$, then

$$(1+t)^{\alpha+\varepsilon}||w(t)||^2 + \int_0^t (1+\tau)^{\alpha+\varepsilon}||w_z(\tau)||^2\,d\tau \leqslant C\left(||w_0||^2 + (\alpha+\varepsilon)\int_0^t (1+\tau)^{\alpha+\varepsilon-1}||w(\tau)||^2\,d\tau\right). \tag{3.25}$$

Next we estimate the second term on the right-hand side of (3.25). For any p, p' with $\frac{1}{p} + \frac{1}{p'} = 1$ and $A > 0$ to be determined, using the Hölder inequality we have

$$\begin{aligned} ||w(\tau)||^2 &= \int_{\mathbb{R}} w^{2(\frac{1}{p}+\frac{1}{p'})}(\tau)\langle z - z^* \rangle^{A-A}\,dz \\ &\leqslant \left(\int_{\mathbb{R}} \langle z - z^* \rangle^{Ap} w^2\,dz\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \langle z - z^* \rangle^{-Ap'} w^2\,dz\right)^{\frac{1}{p'}} \\ &= (|w(\tau)|^2_{Ap})^{\frac{1}{p}}(|w(\tau)|^2_{-Ap'})^{\frac{1}{p'}}. \end{aligned} \tag{3.26}$$

In order to use (3.24) we choose A to satisfy $Ap = \alpha - [\alpha], -Ap' = \alpha - [\alpha] - 1$. This means that $A = \frac{\alpha - [\alpha]}{p}$. One can also show that $A = (\alpha - [\alpha])(1 + [\alpha] - \alpha)$ which implies $p = \frac{1}{1+[\alpha]-\alpha}$. Using (3.26), (3.24) reduces to

$$\begin{aligned} ||w(\tau)||^2 &\leqslant (|w(\tau)|^2_{\alpha-[\alpha]})^{\frac{1}{p}}(|w(\tau)|^2_{\alpha-[\alpha]-1})^{\frac{1}{p'}} \\ &\leqslant C|w_0|_\alpha^{\frac{2}{p}}(1+\tau)^{-\frac{[\alpha]}{p}}(1+\tau)^{-\frac{[\alpha]}{p'}}((1+\tau)^{[\alpha]}|w(\tau)|^2_{\alpha-[\alpha]-1})^{\frac{1}{p'}} \\ &= C|w_0|_\alpha^{\frac{2}{p}}(1+\tau)^{-[\alpha]}((1+\tau)^{[\alpha]}|w(\tau)|^2_{\alpha-[\alpha]-1})^{\frac{1}{p'}}. \end{aligned} \tag{3.27}$$

We want to estimate the second term on the right-hand side of (3.25) by inserting (3.27). Thus we use the Hölder inequality and (3.24), we get using $p = \frac{1}{1+[\alpha]-\alpha}$

$$\begin{aligned} &\int_0^t (1+\tau)^{\alpha+\varepsilon-1}||w(\tau)||^2\,d\tau \quad \text{by (3.27)} \\ &\leqslant C|w_0|_\alpha^{\frac{2}{p}} \int_0^t (1+\tau)^{\alpha+\varepsilon-1-[\alpha]}((1+\tau)^{[\alpha]}|w(\tau)|^2_{\alpha-[\alpha]-1})^{\frac{1}{p'}}\,d\tau \end{aligned}$$

$$
\begin{aligned}
&\leqslant C|w_0|_\alpha^{\frac{2}{p}}\left(\int_0^t (1+\tau)^{p(\alpha+\varepsilon-1-[\alpha])}\,d\tau\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^t (1+\tau)^{[\alpha]}|w(\tau)|^2_{\alpha-[\alpha]-1}\,d\tau\right)^{\frac{1}{p'}} \quad \text{by} \quad (3.24) \\
&\leqslant C|w_0|_\alpha^2\left(\int_0^t (1+\tau)^{\alpha+\varepsilon-1-[\alpha]}\,d\tau\right)^{\frac{1}{p}} \\
&\leqslant C|w_0|_\alpha^2\left(\frac{(1+t)^{p(\alpha+\varepsilon-1-[\alpha])+1}-1}{p(\alpha+\varepsilon-1-[\alpha])+1}\right)^{\frac{1}{p}} \\
&= C|w_0|_\alpha^2\left(\frac{(1+t)^{p\varepsilon}-1}{p\varepsilon}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Thereby we have

$$\int_0^t (1+\tau)^{\alpha+\varepsilon-1}||w(\tau)||^2\,d\tau \leqslant C_\varepsilon|w_0|_\alpha^2(1+t)^\varepsilon$$

with $C_\varepsilon \to \infty$, as $\varepsilon \to 0$. Therefore, inserting this estimate and taking a new C_ε we have

$$(1+t)^{\alpha+\varepsilon}||w(t)||^2 + \int_0^t (1+\tau)^{\alpha+\varepsilon}||w_z(\tau)||^2\,d\tau \leqslant C_\varepsilon|w_0|_\alpha^2(1+t)^\varepsilon.$$

This proves Lemma 3.7. $\square$

Lemma 3.8. *Let* $0 \leqslant k \leqslant \alpha$. *Then*

$$(1+t)^k|w(t)|^2_{\alpha-k} + (1+t)^{-\varepsilon}\int_0^t (1+\tau)^{k+\varepsilon}|w_z(\tau)|^2_{\alpha-k}\,d\tau \leqslant C_\varepsilon|w_0|_\alpha^2$$

where $C_\varepsilon \to \infty$ *as* $\varepsilon \to 0$.

Proof. Note that if $\alpha = k$ then Lemma 3.8 is the same as Lemma 3.7. Hence, we consider the case $k < \alpha$. In (3.16) we take $\beta = \alpha - k$, $\gamma = k + \varepsilon < \alpha$. Then

$$
\begin{aligned}
&(1+t)^{k+\varepsilon}|w(t)|_{\alpha-k}^2+\int_0^t(\alpha-k)(1+\tau)^{k+\varepsilon}|w(\tau)|_{\alpha-k-1}^2\,d\tau\\
&\quad+\int_0^t(1+\tau)^{k+\varepsilon}|w_z|_{\alpha-k}^2\,d\tau\\
&\leqslant C\left\{|w_0|_{\alpha-k}^2+\int_0^t(k+\varepsilon)(1+\tau)^{k+\varepsilon-1}|w(\tau)|_{\alpha-k}^2\,d\tau\right.\\
&\quad\left.+\int_0^t(\alpha-k)(1+\tau)^{k+\varepsilon}||w_z(\tau)||^2\,d\tau\right\}.
\end{aligned}
\tag{3.28}
$$

From Lemma 3.7 we have

$$
\int_0^t(1+\tau)^{k+\varepsilon}||w_z(\tau)||^2\,d\tau\leqslant C|w_0|_\alpha(1+t)^\varepsilon.
\tag{3.29}
$$

Next, we estimate the second term on the right-hand side in (3.28). Here we need to use the following inequalities from (3.17)

$$
(1+\tau)^{[k]}|w(t)|_{\alpha-[k]}^2\leqslant C|w_0|_\alpha^2
$$

and

$$
\int_0^t(1+\tau)^{[k]}|w(\tau)|_{\alpha-[k]-1}^2\,d\tau\leqslant C|w_0|_\alpha^2.
$$

Now we choose $p=\frac{1}{1+[k]-k}$, $A=\frac{k-[k]}{p}$. A similar procedure as in the proof of Lemma 3.7 yields

$$
(k+\varepsilon)\int_0^t(1+\tau)^{k+\varepsilon-1}|w(\tau)|_{\alpha-k}^2\,d\tau\leqslant C_\varepsilon|w_0|_\alpha^2(1+t)^\varepsilon.
$$

This proves Lemma 3.8. $\square$

From Lemma 3.8 it follows that

$$
|w(t)|_{\alpha-k}\leqslant\sqrt{C_\varepsilon}(1+t)^{-k/2}|w_0|_\alpha.
$$

Taking any $\varepsilon=\varepsilon_0>0$ so that C_ε is a fixed constant, we obtain estimate (3.1) with $C=\sqrt{C_{\varepsilon_0}}$. This completes the proof of Theorem 3.1. $\square$

4. Decay rate of first derivatives

Now we turn to estimate the decay rate for the derivative of w.

Theorem 4.1. *Let w satisfy the same conditions as in Theorem* 3.1. *If $w_z(\cdot,0)\in L^2_{\alpha+1}(\mathbb{R})$, then the following estimate holds*:

$$|w_z(t)|_{\alpha+1-k}\leqslant C[|w_0|_\alpha+|w_z(\cdot,0)|_{\alpha+1}](1+t)^{-\frac{k}{2}} \tag{4.1}$$

for $0\leqslant k\leqslant\alpha$. In case $w_z(\cdot,0)\in L^2_\alpha(\mathbb{R})$, then (4.1) *holds only for $1\leqslant k\leqslant\alpha$.*

The proof of Theorem 4.1 is very technical and was broken up into a number of lemmas. For the proof we consider the following.

We assume that

$$u(x,t)=\phi(z)+w_z(z,t),\quad z=x-st$$

is the solution of Eq. (1.1). Letting $v=w_z(z,t)$, we can get from (1.1)–(1.3) with changing variables from (x,t) to (z,t)

$$v_t+(Q(\phi+v)-Q(\phi))_z+v_{zzz}-\mu v_{zz}=0. \tag{4.2}$$

Multiplying Eq. (4.2) by $2\langle z_M\rangle^\beta v$, where $\langle z_M\rangle=\sqrt{M^2+z^2}$ with M a positive constant to be appropriately chosen below. Integrating the result over $\mathbb{R}$ we have

$$\begin{aligned}2\int_{\mathbb{R}}\langle z_M\rangle^\beta vv_t\,dz+2\int_{\mathbb{R}}\langle z_M\rangle^\beta v(Q(\phi+v)-Q(\phi))_z\,dz\\ +2\int_{\mathbb{R}}\langle z_M\rangle^\beta vv_{zzz}\,dz-2\mu\int_{\mathbb{R}}\langle z_M\rangle^\beta vv_{zz}\,dz=0.\end{aligned} \tag{4.3}$$

After several integrations by parts, we have

$$2\int_{\mathbb{R}}\langle z_M\rangle^\beta vv_t\,dz=\left[\int_{\mathbb{R}}\langle z_M\rangle^\beta v^2\,dz\right]_t, \tag{4.4}$$

$$2\int_{\mathbb{R}}\langle z_M\rangle^\beta vv_{zzz}\,dz=-\int_{\mathbb{R}}\langle z_M\rangle^\beta_{zzz}v^2\,dz+3\int_{\mathbb{R}}\langle z_M\rangle^\beta_z v_z^2\,dz \tag{4.5}$$

and

$$-2\mu\int_{\mathbb{R}}\langle z_M\rangle^\beta vv_{zz}\,dz=-\mu\int_{\mathbb{R}}\langle z_M\rangle^\beta_{zz}v^2\,dz+2\mu\int_{\mathbb{R}}\langle z_M\rangle^\beta v_z^2\,dz. \tag{4.6}$$

In order to bound the second term in (4.3), which is the key in the proof of Theorem 4.1, we define

$$[Q]:=Q(\phi+v)-Q(\phi)\quad\text{and}\quad B:=\int_\phi^{\phi+v}Q(y)\,dy-Q(\phi)v.$$

We have the following lemma.

Lemma 4.2. *For $t > 0$, it holds*

$$\left| \int_{\mathbb{R}} \langle z_M \rangle^{\beta} v(Q(\phi+v) - Q(\phi))_{dz} \right| \leqslant C \int_{\mathbb{R}} \langle z_M \rangle^{\beta-1} v^2 \, dz. \tag{4.7}$$

Proof. Since

$$\begin{aligned} B_z &= Q(\phi+v)(\phi_z+v_z) - Q(\phi)\phi_z - Q'(\phi)\phi_z v - Q(\phi)v_z \\ &= [Q](\phi_z+v_z) - Q'(\phi)\phi_z v, \end{aligned}$$

we have

$$v_z[Q] = B_z - ([Q] - Q'(\phi)v)\phi_z = B_z - \tfrac{1}{2}Q''(\xi)v^2\phi_z$$

with ξ being between ϕ and $\phi+v$, and

$$\begin{aligned} &\langle z_M \rangle^{\beta} v(Q(\phi+v) - Q(\phi))_z \\ &\quad = (\langle z_M \rangle^{\beta} v[Q])_z \\ &\qquad - v_z[Q]\langle z_M \rangle^{\beta} - \langle z_M \rangle^{\beta}_z v[Q] \\ &\quad = (\langle z_M \rangle^{\beta} v[Q])_z - \langle z_M \rangle^{\beta}(B_z - \tfrac{1}{2}Q''(\xi)v^2\phi_z) \\ &\qquad - \langle z_M \rangle^{\beta}_z v[Q] \\ &\quad = (\langle z_M \rangle^{\beta} v[Q] - \langle z_M \rangle^{\beta} B)_z + \langle z_M \rangle^{\beta}_z (B - v[Q]) \\ &\qquad + \tfrac{1}{2}Q''(\xi)v^2\phi_z\langle z_M \rangle^{\beta}. \end{aligned} \tag{4.8}$$

Using

$$\begin{aligned} B - v[Q] &= \int_{\phi}^{\phi+v} Q(y)\, dy - Q(\phi+v)v \\ &= (Q(\phi+\theta v) - Q(\phi+v))v \\ &= (\theta-1)v^2 Q'(\xi'), \end{aligned}$$

where $0 < \theta < 1$ and ξ' is between $\phi+\theta v$ and $\phi+v$, (4.8) reduces to

$$\begin{aligned} &\langle z_M \rangle^{\beta} v[Q(\phi+v) - Q(\phi)]_z \\ &\quad = (\langle z_M \rangle^{\beta} v[Q] - \langle z_M \rangle^{\beta} B)_z + \langle z_M \rangle^{\beta}_z (\theta-1)v^2 Q'(\xi') \\ &\qquad + \tfrac{1}{2}Q''(\xi)v^2\phi_z\langle z_M \rangle^{\beta}. \end{aligned}$$

Hence, since ϕ, ϕ_z and $v = w_z$ are bounded, see Corollary 2.3.2 in Chapter 2 of [11], we have

$$\begin{aligned}&\left|\int_{\mathbb{R}} \langle z_M \rangle^{\beta} v(Q(\phi+v)-Q(\phi))_z\,dz\right|\\&\quad\leqslant C\left(\int_{\mathbb{R}} \langle z_M \rangle_z^{\beta} v^2\,dz+\int_{\mathbb{R}} |\langle z_M \rangle^{\beta}\phi_z| v^2\,dz\right)\\&\quad\leqslant C\int_{\mathbb{R}} \langle z_M \rangle^{\beta-1} v^2\,dz. \end{aligned}\tag{4.9}$$

The lemma is proved. $\square$

By (4.3)–(4.7), we get

$$\begin{aligned}&\left(\int_{\mathbb{R}} \langle z_M \rangle^{\beta} v^2\,dz\right)_t+\int_{\mathbb{R}} (2\mu\langle z_M \rangle^{\beta}+3\langle z_M \rangle_z^{\beta}) v_z^2\,dz\\&\quad\leqslant C\int_{\mathbb{R}} \langle z_M \rangle^{\beta-1} v^2\,dz+\int_{\mathbb{R}} \langle z_M \rangle_{zzz}^{\beta} v^2\,dz+\mu\int_{\mathbb{R}} \langle z_M \rangle_{zz}^{\beta} v^2\,dz. \end{aligned}\tag{4.10}$$

Further, the following lemma is true.

Lemma 4.3. *For $\beta \leqslant \alpha+1$ and $M > \frac{3(\alpha+1)}{2\mu}$, we have the estimate*

$$(|v(t)|_{\beta}^2)_t+|v_z(t)|_{\beta}^2 \leqslant C|v(t)|_{\beta-1}^2. \tag{4.11}$$

Proof. Using $|z| < \langle z_M \rangle$ one gets

$$\begin{aligned}|\langle z_M \rangle_{zz}^{\beta}| &= |\beta\langle z_M \rangle^{\beta-2}+\beta(\beta-2)z^2\langle z_M \rangle^{\beta-4}|\\&\leqslant C_\alpha\langle z_M \rangle^{\beta-2} \leqslant C_\alpha\langle z_M \rangle^{\beta-1}\end{aligned}$$

and

$$\begin{aligned}|\langle z_M \rangle_{zzz}^{\beta}| &= |3\beta(\beta-2)z\langle z_M \rangle^{\beta-4}+\beta(\beta-2)(\beta-4)z^3\langle z_M \rangle^{\beta-6}|\\&\leqslant C_\alpha\langle z_M \rangle^{\beta-3}\\&\leqslant C_\alpha\langle z_M \rangle^{\beta-1}.\end{aligned}$$

In addition, by choosing $M > \frac{3(\alpha+1)}{2\mu}$, and β satisfying $\beta \leqslant \alpha+1$, one gets

$$\begin{aligned}2\mu\langle z_M\rangle^\beta + 3\langle z_M\rangle^\beta_z &= 2\mu\langle z_M\rangle^\beta + 3\beta z\langle z_M\rangle^{\beta-2}\\ &= \langle z_M\rangle^{\beta-2}(2\mu z^2 + 3\beta z + 2\mu M^2)\\ &\geqslant \mu\langle z_M\rangle^\beta.\end{aligned}$$

Substituting these three inequalities into (4.10), and noting that for any fixed M there always exists a positive constant C such that

$$C^{-1}\langle z - z^*\rangle \leqslant \langle z_M\rangle \leqslant C\langle z - z^*\rangle,$$

we immediately get (4.11). □

Lemma 4.4. *For* $1\leqslant k\leqslant\alpha$ *with fixed* α, *we have the estimate*

$$\begin{aligned}&(1+t)^{k+\varepsilon}|v(t)|^2_{\alpha+1-k} + \int_0^t (1+\tau)^{k+\varepsilon}|v_z(t)|^2_{\alpha+1-k}\,d\tau\\ &\quad\leqslant C(|w_0|^2_\alpha + |v_0|^2_\alpha)(1+t)^\varepsilon.\end{aligned}\tag{4.12}$$

Proof. Let δ be a parameter. Multiplying (4.11) by $(1+t)^\delta$, one gets

$$[(1+t)^\delta|v|^2_\beta]_t - \delta(1+t)^{\delta-1}|v|^2_\beta + (1+t)^\delta|v_z|^2_\beta \leqslant C(1+t)^\delta|v|^2_{\beta-1}.$$

Integrating this result with respect to time we get

$$\begin{aligned}(1+t)^\delta|v|^2_\beta + \int_0^t(1+\tau)^\delta|v_z|^2_\beta\,d\tau \leqslant C\bigg(&|v_0|^2_\beta + \int_0^t(1+\tau)^\delta|v|^2_{\beta-1}\,d\tau\\ &+\int_0^t\delta(1+\tau)^{\delta-1}|v|^2_\beta\,d\tau\bigg).\end{aligned}\tag{4.13}$$

Let $1\leqslant\beta\leqslant\alpha$, $\beta-1=\alpha-k\geqslant 0$, and $\delta = k+\varepsilon$ in (4.13) we get

$$\begin{aligned}&(1+t)^{k+\varepsilon}|v|^2_{\alpha+1-k} + \int_0^t(1+\tau)^{k+\varepsilon}|v_z|^2_{\alpha+1-k}\,d\tau\\ &\quad\leqslant C\bigg(|v_0|^2_{\alpha+1-k} + \int_0^t(1+\tau)^{k+\varepsilon}|v|^2_{\alpha-k}\,d\tau + \int_0^t(k+\varepsilon)(1+\tau)^{k+\varepsilon-1}|v|^2_{\alpha+1-k}\,d\tau\bigg).\end{aligned}$$

From Lemma 3.8, we have for $v = w_z$

$$\int_0^t(1+\tau)^{k+\varepsilon}|v|^2_{\alpha-k}\,d\tau \leqslant C(1+t)^\varepsilon|w_0|^2_\alpha \quad \text{for } 0\leqslant k\leqslant\alpha \tag{4.14}$$

and

$$\int_0^t (1+\tau)^{k+\varepsilon-1}|v|^2_{\alpha+1-k}\,d\tau \leqslant C(1+t)^{\varepsilon}|w_0|^2_\alpha \quad \text{for } 1\leqslant k\leqslant \alpha. \tag{4.15}$$

Therefore, (4.12) has been proved. □

Lemma 4.5. *If* $v_0 \in L^2_{\alpha+1}$, *then we have the estimate*

$$|v(t)|^2_{\alpha+1} + \int_0^t |v_z(\tau)|^2_{\alpha+1}\,d\tau \leqslant C(|w_0|^2_\alpha + |v_0|^2_{\alpha+1}). \tag{4.16}$$

Proof. By (3.17) with $k=0$ we get

$$\int_0^t |v(\tau)|^2_\alpha\,d\tau \leqslant C|w_0|^2_\alpha. \tag{4.17}$$

If $v_0 \in L^2_{\alpha+1}$, then (4.13) with $\delta = 0$ and $\beta = \alpha+1$ gives

$$\begin{aligned} |v(t)|^2_{\alpha+1} + \int_0^t |v_z(\tau)|^2_{\alpha+1}\,d\tau &\leqslant C\left(|v_0|^2_{\alpha+1} + \int_0^t |v(\tau)|^2_\alpha\,d\tau\right) \\ &\leqslant C(|w_0|^2_\alpha + |v_0|^2_{\alpha+1}), \end{aligned} \tag{4.18}$$

which proves (4.16). □

Lemma 4.6. *For* $0<k<1$, *we may estimate*

$$\begin{aligned} &(1+t)^{k+\varepsilon}|v(t)|^2_{\alpha+1-k} + \int_0^t (1+\tau)^{k+\varepsilon}|v_z(t)|^2_{\alpha+1-k}\,d\tau \\ &\quad \leqslant C[|w_0|^2_\alpha + |v_0|^2_{\alpha+1}](1+t)^{\varepsilon}. \end{aligned} \tag{4.19}$$

Proof. For $0<k<1$, i.e. for $\alpha<\beta<\alpha+1$ with $\beta = \alpha+1-k$ and $\delta = k+\varepsilon$, inequality (4.13) implies

$$\begin{aligned} &(1+t)^{k+\varepsilon}|v|^2_{\alpha+1-k} + \int_0^t (1+\tau)^{k+\varepsilon}|v_z|^2_{\alpha+1-k}\,d\tau \\ &\quad \leqslant C\left(|v_0|^2_{\alpha+1} + \int_0^t (1+\tau)^{k+\varepsilon}|v|^2_{\alpha-k}\,d\tau\right) \\ &\qquad + \int_0^t (k+\varepsilon)(1+\tau)^{k+\varepsilon-1}|v|^2_{\alpha+k-1}\,d\tau \\ &\quad \leqslant C\left((1+t)^{\varepsilon}|w_0|^2_\alpha + |v_0|^2_{\alpha+1} + \int_0^t (1+\tau)^{k+\varepsilon-1}|v|^2_{\alpha+1-k}\,d\tau\right), \end{aligned}$$

where we have used (4.14). Next we estimate

$$\int_0^t (1+\tau)^{k+\varepsilon-1}|v|^2_{\alpha+1-k}\,d\tau$$

for $0<k<1$, in the same way as before. In fact, for $\frac{1}{q}+\frac{1}{q'}=1$, $B+B'=\alpha+1-k, qB=\alpha+1, q'B'=\alpha$, we get

$$\begin{aligned}
&\int_0^t (1+\tau)^{k+\varepsilon-1}|v|^2_{\alpha+1-k}\,d\tau \\
&\quad= \int_0^t (1+\tau)^{k+\varepsilon-1}\int_{\mathbb{R}} \langle z_M \rangle^{\alpha+1-k} v^2\,dz\,d\tau \\
&\quad= \int_0^t (1+\tau)^{k+\varepsilon-1}\int_{\mathbb{R}} \langle z_M \rangle^{B+B'} v^2\,dz\,d\tau \\
&\quad\leqslant \int_0^t (1+\tau)^{k+\varepsilon-1}\left(\int_{\mathbb{R}} \langle z_M \rangle^{qB} v^2\,dz\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}} \langle z_M \rangle^{q'B'} v^2\,dz\right)^{\frac{1}{q'}}d\tau \\
&\quad= \int_0^t (1+\tau)^{k+\varepsilon-1}\left(\int_{\mathbb{R}} \langle z_M \rangle^{\alpha+1} v^2\,dz\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}} \langle z_M \rangle^{\alpha} v^2\,dz\right)^{\frac{1}{q'}}d\tau \\
&\quad= \int_0^t (1+\tau)^{k+\varepsilon-1}(|v|^2_{\alpha+1})^{\frac{1}{q}}(|v|^2_{\alpha})^{\frac{1}{q'}}\,d\tau.
\end{aligned}$$

Using (4.16), (4.17) and Hölder inequality, we have

$$\begin{aligned}
&\int_0^t (1+\tau)^{k+\varepsilon-1}|v|^2_{\alpha+1-k}\,d\tau \\
&\quad= \int_0^t (1+\tau)^{k+\varepsilon-1}(|v|^2_{\alpha+1})^{\frac{1}{q}}(|v|^2_{\alpha})^{\frac{1}{q'}}\,d\tau \\
&\quad\leqslant C(|w_0|^2_\alpha+|v_0|^2_{\alpha+1})^{\frac{1}{q}}\int_0^t (1+\tau)^{k+\varepsilon-1}(|v|^2_\alpha)^{\frac{1}{q'}}\,d\tau \\
&\quad\leqslant C(|w_0|^2_\alpha+|v_0|^2_{\alpha+1})^{\frac{1}{q}}\left(\int_0^t (1+\tau)^{q(k+\varepsilon-1)}\,d\tau\right)^{\frac{1}{q}}\left(\int_0^t |v|^2_\alpha\,d\tau\right)^{\frac{1}{q'}} \\
&\quad\leqslant C(|w_0|^2_\alpha+|v_0|^2_{\alpha+1})^{\frac{1}{q}}(|w_0|^2_\alpha)^{\frac{1}{q'}}\left(\frac{(1+t)^{q(k+\varepsilon-1)+1}}{q(k+\varepsilon-1)+1}\right)^{\frac{1}{q}} \\
&\quad\leqslant C(|w_0|^2_\alpha+|v_0|^2_{\alpha+1})^{\frac{1}{q}}(|w_0|^2_\alpha)^{\frac{1}{q'}}(1+t)^{\varepsilon}
\end{aligned}$$

where $q=-\frac{1}{k-1}$, $0<k<1$. Combining the estimates obtained, we have proved (4.19). □

At this stage we can conclude Theorem 4.1 by combining Lemma 4.4 with Lemma 4.6.

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